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# NOVEL APPROACHES FOR s-CONVEX FUNCTIONS VIA CAPUTO-FABRIZIO FRACTIONAL INTEGRALS

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**Abstract** There are several studies in the literature with the main motivation of obtaining new and general inequalities with the help of the Caputo-Fabrizio fractional integral operator, which attracts the attention of many researchers as an important concept in fractional analysis. In this study, new Hadamard type inequalities for $s-$convex functions are presented. The findings were obtained by the properties of the class of the function, the structure of the operator and the basic analysis methods.

**Keywords**: Caputo-Fabrizio integral operators, $s-$convexity, Hölder inequality, Young inequality.

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## 1. Introduction

Inequality theory is a field in which many researchers work, with new findings that can be given applications in many disciplines such as mathematical analysis, statistics, approximation theory and numerical analysis together with convex functions. Although the concept of convex function is a concept intertwined with inequalities by definition, it has also formed the main motivation of many researches with its aesthetic structure, features and different types. Let's start with the definition of this important class of functions.

**Definition 1.1.** *A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

*for all* $x, y \in I,\ \lambda \in [0,1]$.

The Hermite-Hadamard inequality, a famous inequality made using convex functions, produces lower and upper bounds for the mean value of a convex function. Let us now recall the Hermite-Hadamard inequality.

The function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, for $a, b \in I$ and $f \in L_1[a, b]$ then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

**Definition 1.2.** ([3]) *A function $f: [0, \infty) \to \mathbb{R}$ is said to be $s$ −convex in the first sense if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

*for all* $x, y \in [0, \infty)$, $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$.

Fractional analysis has its origins as old as classical analysis and has recently become a trending topic that has increased its effectiveness. Fractional analysis, which is basically built on fractional derivatives and integrals, has been the focus of attention of many researchers due to its advantages over classical analysis in explaining physical phenomena.

The fact that the derivative operator does not have a single definition in fractional analysis comes to the fore in obtaining stable solutions for the solution of differential equations and real world problems. The new fractional derivative operators and associated integral operators introduced in the field of fractional analysis differ from each other in terms of their various properties. It is known that derivative operators differ according to their usage areas, considering the properties of the kernel structure such as locality and singularity, its compatibility with the classical derivative, and the various algebraic properties it provides.

In addition, some of the new operators defined are important because they have the general form and some of them are given in the forms in which the two operators are given together. However, the main motivation point here is the efficiency and productivity of fractional operators in their application areas.

In this study, first of all, a general introduction to the fractional integral operators will be made and then the relations between them will be revealed. By following the basic definitions and historical development, it will be provided to refresh the information by giving important ideas about fractional analysis. We will prove some new integral inequalities of Hadamard type via Caputo-Fabrizio fractional integral operators for $s$ −convex functions.

## 2. Fractional Integral Operators

We will first start by introducing the Riemann-Liouville fractional integral operators, an integral operator that can be considered one of the cornerstones of fractional calculus. The operator in the definition is a concept that is widely used in the field of inequalities as well as fractional differential equations.

**Definition 2.1** *([13]) Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx, \quad t > a \tag{1}$$

and

$$J_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (x-t)^{\alpha-1} f(x)dx, \quad t < b \tag{2}$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.

Here we note that if we set $\alpha = 0$, it is clear that

$$J_{a+}^{\alpha} f(t)\Big|_{\alpha=0} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx \Big|_{\alpha=0} = \int_a^t \frac{(t-x)^{-1}}{(-1)!} f(x)dx = \int_a^t \delta(t-x) f(x)dx = \frac{1}{2} f(t)$$

In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

In [1], Abdeljawad defined the right-left conformable integrals as followings associated with the conformable derivative:

**Definition 2.2** *([1]) Let $\alpha \in (n, n+1]$, $n = 0,1,2,...$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by*

$$(I_a^{\alpha} f)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

**Definition 2.3** *Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by*

$$({}^{\alpha}I_b f)(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx$$

If we choose $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, hence $(I_{n+1}^{a} f)(t) = (J_{a+}^{n+1} f)(t)$ and $({}^{b}I_{n+1} f)(t) = \left(J_{b-}^{n+1} f\right)(t)$.

After the definition of conformable integrals, the new conformable integrals have a general form, giving a rapid movement to fractional analysis. Due to with this integral, the generalization of various integrals such as Riemann-Liouville fractional integral, Hadamard fractional integrals, generalized fractional integral operators has been revealed. In [9], this integral operator defined by Jarad et al. is given in the following definition.

**Definition 2.4** *([9]) Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$, then the left and right sided fractional conformable integral operators has defined respectively, as follows;*

$${}_{a^+}^{\beta}\mathfrak{J}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt$$

*and*

$$^{\beta}\mathfrak{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left( \frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt.$$

With this newly defined operator, many results in the literature have been generalized and expanded. For example, if we set $\alpha = 1$ in the obtained results are found to yield the same results involving Riemann-Liouville fractional integrals.

Now, the iterations that will lead to the derivation of the Katugampola integral operator will be introduced. Katugampola (see [10] and [11]) has defined the following iterative process in 2011:

$$\int_a^x t_1^{\rho} dt_1 \int_a^{t_1} t_2^{\rho} dt_2 \ldots \int_a^{t_{n-1}} t_n^{\rho} f(t_n) dt_n = \frac{(\rho+1)^{1-n}}{(n-1)!} \int_a^x \left( t^{\rho+1} - \tau^{\rho+1} \right)^{n-1} \tau^{\rho} f(\tau) d\tau,$$

for $n \in \mathbb{N}$. Katugampola's concept of fractional integral, defined in [10] and also in [11] can be remind as followings.

**Definition 2.5.** *([10]) Let $f \in [a,b]$, the left-sided Katugampola fractional integral ${}^{\rho}I_{a^+}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ is defined by*

$$\left( {}^{\rho}I_{a^+}^{\alpha} f \right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} f(t) dt, \qquad x > a$$

the right-sided Katugampola fractional integral ${}^{\rho}I_{b^-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ is defined by

$$\left( {}^{\rho}I_{b^-}^{\alpha} f \right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^{\rho} - x^{\rho})^{1-\alpha}} f(t) dt, \qquad x < b$$

**Remark 2.6.** *If we set $\rho = 1$ then the Katugampola fractional integrals overlapped with the Riemann-Liouville fractional integrals.*

It is noteworthy that all of these operators introduced in the historical development have a kernel structure that includes singularity. In order to eliminate the limitations in the studies in which these operators are used, derivative and associated integral operators with non-singular kernels are given by Caputo and Fabrizio as follows.

***Definition 2.7.*** *([4]) The Caputo-Fabrizio fractional derivative of order $\vartheta$ is as follows*

$$\left( {}^{CF}D_{*}^{\vartheta} f \right)(t) = \frac{1}{1-\vartheta} \int_0^t exp\left( -\vartheta \frac{t-x}{1-\vartheta} \right) f'(x) dx.$$

*and let $0 < \vartheta < 1$. The fractional integral of order $\vartheta$ of a function $f$ is defined by*

$$\left({}^{CF}I^{\vartheta} f\right)(t) = \frac{2(1-\vartheta)}{(2-\vartheta)M(\vartheta)}u(t) + \frac{2\vartheta}{(2-\vartheta)M(\vartheta)}\int_0^t u(s)ds, \qquad t \geq 0$$

.
*where $M(\vartheta)$ is the normalization function.*

## 3. Integral Inequalities via Fractional Integral Operators

In this part of the study, various integral equations that have been brought to the literature with the help of fractional integral operators and some integral inequalities obtained from these equations will be given.

The following integral identiy was proved by Set in 2012.

**Lemma 3.1** ***([17])*** *Let $f:[a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $f' \in L[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$ we have:*

$$\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{b-a}[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)]$$

$$= \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx+(1-t)a)dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx+(1-t)b)dt$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du.$

The author has proved some new Ostrowski type integral inequalities for $s-$convex functions via Riemann-Liouville fractional integrals. The findings of the author are generalizations of the earlier results.

**Theorem 3.2.** ***([17])*** *Let $f:[a,b] \subseteq (0,\infty) \to \mathbb{R}$, be a differentiable mapping on $(a,b)$ with $a<b$ such that $f' \in L[a,b]$. If $|f'|$ is $s-$convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \leq M, x \in a,b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)]\right|$$

$$\leq \frac{M}{b-a}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+s+1}\right]$$

where $\Gamma$ is Euler gamma function.

After proving the inequalities in general form with the Hadamard and Ostrowski type with the help of Rieamnn-Liouville fractional integral operators, the new Hadamard type inequalities containing the products of different types of convex functions were proved with the help of Riemann-Liouville, Katugampola and conformable integrals as follows.

Novel Hermite-Hadamard type inequalities for products of two different kinds of convex functions are presented by Chen and Wu in [5] as follows:

**Theorem 3.3.** *Let $f, g: [a, b] \to \mathbb{R}$, $a,b \in [0,\infty)$, $a<b$ be functions such that and $g$, $fg \in L[a,b]$. If $f$ is convex and nonnegative and $g$ is s-convex on $[a,b]$ for some fixed $s \in [0,1]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+} f(b)g(b) + J^{\alpha}_{b^-} f(a)g(a)\right]$$
$$\leq \left(\frac{1}{\alpha+s+1} + B(\alpha,s+2)\right)M(a,b)$$
$$+\left(B(\alpha+1,s+1) + \frac{1}{(\alpha+s)(\alpha+s+1)}\right)N(a,b),$$

where $M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a)$.

In [18], some similar results have been established for different kinds of convexity via conformable fractional integrals that were defined by Abdeljawad:

**Theorem 3.4.** *Let $f, g: [a, b] \to \mathbb{R}$, be functions with $0 \leq a < b$ and $f, g, fg \in L_1[a,b]$. If $f$ is convex and nonnegative and $g$ is s-convex on $[a,b]$ for some fixed $s \in [0,1]$, then one has the following inequality for conformable fractional integrals:*

$$\frac{1}{(b-a)^{\alpha}}\left[I^{a}_{\alpha} f(b)g(b) + {}^{b}I_{\alpha} f(a)g(a)\right]$$
$$\leq \frac{M(a,b)}{n!}\left[B(n+s+2,\alpha-n) + B(n+1,\alpha-n+s+1)\right]$$
$$+\frac{N(a,b)}{n!}\left[B(n+2,\alpha-n+s) + B(s+n+1,\alpha-n+1)\right]$$

with $\alpha \in (n,n+1]$.
$\left(M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a)\right)$

In [2], by the motivation of the earlier studies, Akdemir et al. have proved some new approaches but now for new conformable fractional integrals that were defined by Jarad et al.:

**Theorem 3.5.** *Let $f,g:[a,b]\to\mathbb{R}$, be functions with $0\le a<b$ and $f,g,fg\in L_1[a,b]$. If $f$ is convex and nonnegative and $g$ is s-convex on $[a,b]$ for some fixed $s\in[0,1]$, then one has the following inequality for new conformable fractional integrals:*

$$\alpha^{\beta-1}\left(\frac{1}{b-a}\right)^{\alpha\beta}\Gamma(\beta)\left[{}^{\beta}\mathfrak{I}_{a^+}^{\alpha}fg(b)+{}^{\beta}\mathfrak{I}_{b^+}^{\alpha}fg(a)\right] \tag{3}$$

$$\le\left[\beta_1(s+2,\alpha)-\beta_1(s+2,\alpha\beta)+\frac{1}{\alpha+s+1}-\frac{1}{\alpha\beta+s+1}\right]M(a,b)$$

$$+\left[\beta_1(2,\alpha+s)-\beta_1(2,\alpha\beta+s)+\beta_1(s+1,\alpha+1)-\beta_1(s+1,\alpha\beta+1)\right]N(a,b)$$

where $\alpha,\beta>0$ and $\beta_1$ is Euler Beta function. $\left(M(a,b)=f(a)g(a)+f(b)g(b),N(a,b)=f(a)g(b)+f(b)g(a)\right)$

In [6], Çelik et al. have discussed some new Hadamard type integral inequalities via Katugampola fractional integrals as followings:

**Theorem 3.6.** *Let $f,g:[a^\rho,b^\rho]\to\mathbb{R}_0^+$, be functions with $0\le a<b$ and $f,g,fg\in X_c^p(a^\rho,b^\rho)$. If $f$ is convex and $g$ is $s$-convex on $[a^\rho,b^\rho]$* for *$\alpha,\rho\in\mathbb{R}^+$,* with $h(u)=u^\rho$ and *for some fixed $s\in 0,1]$, then one has the following inequality for Katugampola fractional integrals:*

$$\frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho-a^\rho)^\alpha}\left[{}^{\rho}\mathcal{J}_{a^+}^{\alpha}[fg\circ h](b)+{}^{\rho}\mathcal{J}_{b^-}^{\alpha}[fg\circ h](a)\right] \tag{4}$$

$$\le M(a^\rho,b^\rho)\frac{1}{\rho}\left[\frac{1}{\alpha+s+1}+B(\alpha,s+2)\right]$$

$$+N(a^\rho,b^\rho)\frac{1}{\rho}\left[B(\alpha+1,s+1)+\frac{1}{(\alpha+s)(\alpha+s+1)}\right],$$

*where*
$M(a^\rho,b^\rho)=f(a^\rho)g(a^\rho)+f(b^\rho)g(b^\rho)$ *and* $N(a^\rho,b^\rho)=f(a^\rho)g(b^\rho)+f(b^\rho)g(a^\rho)$
.
To provide more details related to fractional integral operators and fractional integral inequalities, see the papers [7, 8, 12, 14-16, 19].

## *4.* New Results via Caputo-Fabrizio Fractional Integral Operators

In this section, we will obtain some new Hadamard type integral inequalities for $s-$ convex functions via the Caputo-Fabrizio fractional operators.

**Theorem 4.1.** Let $I \subseteq \mathbb{R}$. Suppose that $f:[a,b] \subseteq I \longrightarrow \mathbb{R}$ is a $s-$convex function in the first sense on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}I_{a^+}^{\alpha} f\right)(k) + \left({}^{CF}I_{b^-}^{\alpha} f\right)(k) \leq \frac{2(1-\alpha)f(k)(s+1) + \alpha(b-a)\left(f(a)+sf(b)\right)}{B(\alpha)(s+1)}.$$

where $B(\alpha) > 0$ is normalization function, $s \in (0,1]$ and $\alpha \in [0,1]$.

**Proof.** By using the definition of $s-$convex function in the first sense, we can write

$$f(ta+(1-t)b) \leq t^s f(a) + (1-t^s)f(b).$$

By integrating both sides of the inequality over $[0,1]$ with respect to $t$, we get

$$\int_0^1 f(ta+(1-t)b)\,dt \leq \int_0^1 t^s f(a)\,dt + \int_0^1 (1-t^s)f(b)\,dt.$$

By changing of the variable as $x = ta + (1-t)b$ and by calculating the integrals in right hand side, we obtain

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+sf(b)}{s+1}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)} f(k)$, we have

$$\frac{2(1-\alpha)}{B(\alpha)} f(k) + \frac{\alpha}{B(\alpha)} \int_a^b f(x)\,dx$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)} f(k) + \frac{\alpha(b-a)}{B(\alpha)} \frac{f(a)+sf(b)}{s+1}.$$

By simplfying the inequality, we get

$$\left(\frac{1-\alpha}{B(\alpha)} f(k) + \frac{\alpha}{B(\alpha)} \int_a^k f(x)\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)} f(k) + \frac{\alpha}{B(\alpha)} \int_k^b f(x)\,dx\right) \leq \frac{2(1-\alpha)}{B(\alpha)} f(k) + \frac{\alpha(b-a)}{B(\alpha)} \frac{f(a)+sf(b)}{s+1}$$

.

This completes the proof.

**Theorem 4.2.** Let $\mathrm{I} \subseteq \mathbb{R}$. Suppose that $f: [a,b] \subseteq \mathrm{I} \longrightarrow \mathbb{R}$ is $s$ −convex function in the first sense on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}I_{a^+}^{\alpha} f\right)(k) + \left({}^{CF}I_{b^-}^{\alpha} f\right)(k)$$
$$\leq \frac{2(1-\alpha) f(k)(ps+1)^{\frac{1}{p}} + \alpha(b-a)\left(f(a) + f(b)(ps)^{\frac{1}{p}}\right)}{B(\alpha)(ps+1)^{\frac{1}{p}}}.$$

where $B(\alpha) > 0$ is normalization function, $s \in (0,1]$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

**Proof.** We will start to the proof with the definition of $s$ −convex function in the first sense, namely

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t^s) f(b).$$

By integrating both sides of the inequality over $[0,1]$ with respect to $t$, we have

$$\int_0^1 f(ta + (1-t)b)\,dt \leq \int_0^1 t^s f(a)\,dt + \int_0^1 (1-t^s) f(b)\,dt.$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq f(a)\left(\left(\int_0^1 t^{ps}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\,dt\right)^{\frac{1}{q}}\right)$$
$$+ f(b)\left(\left(\int_0^1 (1-t)^{ps}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\,dt\right)^{\frac{1}{q}}\right).$$

By using the fact that $|1-(1-t)^{\alpha}|^{\beta} \leq 1-|1-t|^{\alpha\beta}$ for $\alpha > 0, \beta > 1$, we can write

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq f(a)\left(\left(\int_0^1 t^{ps}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\,dt\right)^{\frac{1}{q}}\right)$$
$$+ f(b)\left(\left(\int_0^1 (1-t)^{ps}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\,dt\right)^{\frac{1}{q}}\right).$$

Here, we use the fact that

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq f(a)\left(\left(\frac{1}{ps+1}\right)^{\frac{1}{p}}(1^q)^{\frac{1}{q}}\right)+f(b)\left(\left(\frac{ps}{ps+1}\right)^{\frac{1}{p}}(1^q)^{\frac{1}{q}}\right).$$

By changing of the variable as $x = ta + (1-t)b$, we deduce

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)(ps)^{\frac{1}{p}}}{(ps+1)^{\frac{1}{p}}}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)} f(k)$, we provide

$$\frac{2(1-\alpha)}{B(\alpha)} f(k) + \frac{\alpha}{B(\alpha)} \int_a^b f(x)\, dx \leq \frac{2(1-\alpha)}{B(\alpha)} f(k) + \frac{\alpha(b-a)}{B(\alpha)} \frac{f(a) + f(b)(ps)^{\frac{1}{p}}}{(ps+1)^{\frac{1}{p}}}.$$

By using the definition of the Caputo-Fabrizio integral operators in the resulting inequality, we get the desired result.

.

**Theorem 4.3.** Let $\mathrm{I} \subseteq \mathbb{R}$. Suppose that $f\colon [a,b] \subseteq \mathrm{I} \longrightarrow \mathbb{R}$ is a $s-$convex function in the first sense on $[a,b]$ such taht $f \in L_1[a,b]$. Then ,we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}I_{a^+}^{\alpha} f\right)(k) + \left({}^{CF}I_{b^-}^{\alpha} f\right)(k) \leq \frac{2(1-\alpha) f(k) pq(ps+1) + \alpha(b-a)\left(f(a)(q + p(ps+1) + f(b)\left(pqs + p(ps+1)\right)\right)}{B(\alpha)pq(ps+1)}.$$

where $B(\alpha) > 0$ is normalization function, $s \in (0,1]$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

**Proof.** From the definition of $s-$convex function in the first sense on $[a,b]$, we have

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t^s) f(b).$$

By integrating both sides of the inequality over $[0,1]$ with respect to $t$, we get

$$\int_0^1 f(ta + (1-t)b)\, dt \leq \int_0^1 t^s f(a)\, dt + \int_0^1 (1-t^s) f(b)\, dt.$$

If we apply the Young's inequality to the right-hand side of the inequality, we obtain

$$\int_0^1 |f(ta+(1-t)b)|\,dt \leq f(a)\left(\frac{1}{p}\int_0^1 t^{sp}\,dt+\frac{1}{q}\int_0^1 1^q\,dt\right) + f(b)\left(\frac{1}{p}\int_0^1 (1-t^s)^p\,dt+\frac{1}{q}\int_0^1 1^q\,dt\right).$$

By using the fact that $|1-(1-t)^\alpha|^\beta \leq 1-|1-t|^{\alpha\beta}$ for $\alpha>0, \beta>1$, we can write

$$\int_0^1 |f(ta+(1-t)b)|\,dt \leq f(a)\left(\frac{1}{p}\int_0^1 t^{sp}\,dt+\frac{1}{q}\int_0^1 1^q\,dt\right) + f(b)\left(\frac{1}{p}\int_0^1 (1-t^s)^p\,dt+\frac{1}{q}\int_0^1 1^q\,dt\right).$$

By changing of the variable as $x=ta+(1-t)b$ and by computing the integrals, we obtain

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq f(a)\left(\frac{1}{p(ps+1)}+\frac{1}{q}\right)+f(b)\left(\frac{ps}{p(ps+1)}+\frac{1}{q}\right).$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)(q+p(ps+1)+f(b)\big(pqs+p(ps+1)\big)}{pq(ps+1)}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}f(k)$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^b f(x)\,dx \leq \frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha(b-a)}{B(\alpha)}\frac{f(a)(q+p(ps+1)+f(b)\big(pqs+p(ps+1)\big)}{pq(ps+1)}.$$

By simplfying the inequality, we get the following inequality

$$\left(\frac{1-\alpha}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^k f(x)\,dx\right)+\left(\frac{1-\alpha}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_k^b f(x)\,dx\right)$$
$$\leq\frac{2(1-\alpha)f(k)pq(ps+1)+\alpha(b-a)\Big(f(a)(q+p(ps+1)+f(b)\big(pqs+p(ps+1)\big)\Big)}{B(\alpha)pq(ps+1)}$$

which implies that

$$\left({}^{CF}I_{a^+}^{\alpha}f\right)(k)+\left({}^{CF}I_{b^-}^{\alpha}f\right)(k)$$
$$\leq\frac{2(1-\alpha)f(k)pq(ps+1)+\alpha(b-a)\Big(f(a)(q+p(ps+1)+f(b)\big(pqs+p(ps+1)\big)\Big)}{B(\alpha)pq(ps+1)}.$$

This completes the proof.

**Remark 4.4.** If we set $s=1$ in the main findings, we obtain new estimations for convex functions via Caputo-Fabrizio fractional integrals.

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